

Scattering problem for the local parametric resonance equation ^{*}

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Abstract

In this paper we present the solution of local parametric resonance equation in terms of parabolic cylinder functions and solve the scattering problem.

1 Introduction

In this work we present a solution of scattering problem for an equation

$$i\frac{dw}{dt} + tw + a\bar{w} = 0. \quad (1)$$

here \bar{w} is a complex conjugate function of w , a is a real constant.

If we use the constant coefficient T instead of t in the second term of equation (1) then such equation describes a well-known parametric resonance phenomenon [1]. In our case equation (1) determines an amplitude of oscillating solution for a local parametric resonance. The most interesting problem for applications is a scattering problem for (1). The statement of the problem is as follows: we will determine the asymptotic behavior of solution for (1) when $t \rightarrow \infty$ with the given asymptotic behavior when $t \rightarrow -\infty$.

Firstly the local nonparametric resonance was studied in [2], [3]. It was shown that the passage through the local resonance is described by the Fresnel integral. The analogous formula for the parametric resonance was unknown up to now. Here we present it.

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Well-known handbooks contain the explicit expression neither for a solution of system for real and imaginary parts of solution $w = x + iy$

$$\begin{aligned}\frac{dx}{dt} &= (a - t)y, \\ \frac{dy}{dt} &= (a + t)x.\end{aligned}\tag{2}$$

nor for the second order differential equation that is equivalent to this system

$$x'' + (t^2 - a^2)x + \frac{x'}{t - a} = 0.\tag{3}$$

In this paper we present the formula for the solution of system (2) and close this gap.

2 Result

Theorem 1 *The solution of equation (1) is represented by parabolic cylinder functions*

$$\begin{aligned}w(t, a) &= \frac{1}{2}e^{-i\frac{3\pi}{4} - \frac{3\pi a^2}{8}} a \overline{U} D_z(e^{\frac{i\pi}{4}} \sqrt{2}t) + \\ &\left(e^{\frac{\pi a^2}{8}} U + 2e^{i\frac{3\pi}{4} + \frac{\pi a^2}{8}} \frac{\sqrt{2\pi}}{a\Gamma(-i\frac{a^2}{2})} \overline{U} \right) D_{-z-1}(e^{\frac{i5\pi}{4}} \sqrt{2}t).\end{aligned}\tag{4}$$

Here $z = i\frac{a^2}{2} - 1$, the function $D_n(\zeta)$ is the parabolic cylinder function [6], the parameter U is an arbitrary complex constant.

Corollary. Let the solution of (1) be

$$w = V e^{i(\frac{t^2}{2} - \frac{a^2}{2} \ln(-t))} (1 + O(t^{-1})), \quad \forall V \in \mathbf{C}, \quad t \rightarrow -\infty,\tag{5}$$

then this solution has the following form as $t \rightarrow \infty$

$$w = \left(e^{\frac{a^2\pi}{2}} V + \frac{e^{i\frac{\pi - 2a^2 \ln(2)}{4} + \frac{\pi a^2}{4}} a \sqrt{\pi} \overline{V}}{\Gamma(1 - i\frac{a^2}{2})} \right) e^{i(\frac{t^2}{2} - \frac{a^2}{2} \ln(t))} (1 + O(t^{-1})).\tag{6}$$

Formulas (5) and (6) give the connection formulas for solutions of equation (1).

3 Proof of Theorem 1

The proof of Theorem 1 consists of three steps. The first step shows that the equation of parabolic cylinder is a differential consequence of equation (1). It yields that the solution of (1) can be represented as a linear combination of known solutions of parabolic cylinder equation. The second step contains the construction of WKB-asymptotic representation for the solution of (1) when $t \rightarrow -\infty$. The final step allows us to obtain formulas that represent the solution of (1) by linear independent solutions of parabolic cylinder equation.

3.1 Reduction to the parabolic cylinder equation

Lemma 1 *The solution of equation (1) is a combination of linearly independent solutions of parabolic cylinder equation with special values of constants c_1 and c_2 :*

$$w(t, a) = c_1 D_n(e^{i\frac{\pi}{4}}\sqrt{2}t) + c_2 D_{-n-1}(e^{i\frac{3\pi}{4}}\sqrt{2}t), \quad (7)$$

where $n = -1 + ia^2/2$.

To prove Lemma 1 we differentiate equation (1) with respect to t . It yields the differential equation of the second order. We change the derivative \overline{w}' according to the complex conjugated equation of (1) and \overline{w} according to equation (1). It yields the following equation

$$w'' - (i + a^2 - t^2)w = 0. \quad (8)$$

Denote $W(\zeta) = w(t, a)$, where $\zeta = e^{i\frac{\pi}{4}}\sqrt{2}t$, $n = -1 + ia^2/2$. This substitution leads to the standard form of parabolic cylinder equation for function $W(\zeta)$

$$\frac{d^2 W}{d\zeta^2} + (n + \frac{1}{2} - \frac{1}{4}\zeta^2)W = 0.$$

A general solution of this equation can be represented as a combination of linear independent solutions

$$W = c_1 D_n(\zeta) + c_2 D_{-n-1}(i\zeta).$$

The solution of equation (1) satisfies equation (8) as well and can be represented in form (7). Lemma 1 proved.

3.2 WKB-asymptotic representation

Lemma 2 *When $t \rightarrow -\infty$ the solution of equation (1) has the form:*

$$w = e^{i(\frac{1}{2}t^2 - \frac{1}{2}a^2 \ln(-t))} U + \frac{1}{t} e^{-i(\frac{1}{2}t^2 - \frac{1}{2}a^2 \ln(-t))} a \overline{U} + O(t^{-2}). \quad (9)$$

The proof of Lemma 2 can be obtained by the standard way. We construct the formal asymptotic solution of the form

$$w = e^{i(\Omega(t)+\omega \ln(-t))} \sum_{k=0}^{\infty} t^{-2k} \phi_k + e^{-i(\Omega(t)+\omega \ln(-t))} \sum_{k=1}^{\infty} t^{-2k+1} \psi_k.$$

Substitute this formula into equation (1) and gather the terms of the same order with respect to t . It yields the recurrent sequence of equations for $\Omega(t), \omega$ and coefficients of asymptotic solution.

Relations of order t give

$$(-\Omega' + t)\phi_0 e^{i(\Omega(t)+\omega \ln(-t))} = 0,$$

and hence $\Omega = t^2/2$.

Relations of order t^0 have the form

$$(2\psi_1 + a\overline{\phi_0})e^{-i(\Omega(t)+\omega \ln(-t))} = 0,$$

and hence $\psi_1 = -a\overline{\phi_0}/2$.

Relations of order t^{-1} have the form

$$(-\omega - \frac{1}{2}|a|^2)\phi_0 e^{i(\Omega(t)+\omega \ln(-t))} = 0,$$

and give $\omega = -|a|^2/2$.

Denote $\phi_0 = U$, where $U \in \mathbf{C}$ and obtain the leading-order terms of the asymptotic representation of (9). The higher order terms are evaluated in the same manner. The justification of these asymptotic formulas is realized by the standard way, see [7]. Lemma 2 proved.

3.3 Matching of asymptotic representations

In this section we match asymptotic formula (9) and formula of the solution (7). To obtain this result we use well-known formulas for asymptotic representation of parabolic cylinder functions [6]. Here we present these formulas for reference

$$\begin{aligned} D_n(\zeta) &= e^{-\frac{1}{4}\zeta^2} \zeta^n (1 - O(\zeta^{-2})), & |\arg(\zeta)| < \frac{3}{4}\pi; \\ D_n(\zeta) &= e^{-\frac{1}{4}\zeta^2} \zeta^n (1 - O(\zeta^{-2})) - \\ &\quad \frac{\sqrt{2\pi}}{\Gamma(-n)} e^{in\pi} \zeta^{-n-1} (1 + O(\zeta^{-2})), & \frac{1}{4}\pi < \arg(\zeta) < \frac{5}{4}\pi; \\ D_n(\zeta) &= e^{-\frac{1}{4}\zeta^2} \zeta^n (1 - O(\zeta^{-2})) - \\ &\quad \frac{\sqrt{2\pi}}{\Gamma(-n)} e^{-in\pi} \zeta^{-n-1} (1 + O(\zeta^{-2})), & -\frac{5}{4}\pi < \arg(\zeta) < -\frac{1}{4}\pi. \end{aligned} \tag{10}$$

Substitute asymptotic representation (9) into the left-hand side of (7) and asymptotic representation (10) of parabolic cylinder functions into the right-hand side of (7). Gathering the terms of the same order with respect to t gives the statement of Theorem 1. Theorem 1 proved.

The corollary of the Theorem 1 is obtained from the general solution (4) of equation (8) and known asymptotic representation (10) for parabolic cylinder functions.

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